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On the Generic Nonconvergence of  
Bayesian Actions and Beliefs

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*Mark Feldman*





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On the Generic Nonconvergence of Bayesian  
Actions and Beliefs

Mark Feldman

Department of Economics  
University of Illinois at Urbana-Champaign

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## ABSTRACT

Suppose  $Y_n$  is a sequence of i.i.d. random variables taking values in  $Y$ , a complete, separable, non-finite metric space. The probability law indexed by  $\theta \in \Theta$ , is unknown to a Bayesian statistician with prior  $\mu$ , observing this process. Generalizing Freedman [1965, *Annals of Mathematical Statistics*], we show that “generically” (i.e., for a residual family of  $(\theta, \mu)$  pairs) the posterior beliefs do not weakly converge to a point-mass at the “true”  $\theta$ . Furthermore, for every open set  $G \subset \Theta$ , generically, the Bayesian will attach probability arbitrarily close to one to  $G$  infinitely often.

The above result is applied to a two-armed bandit problem with geometric discounting where arm  $k$  yields an outcome in a complete, separable metric space  $Y_k$ . If the infimum of the possible rewards from playing arm  $k$  is less than the infimum from playing arm  $k'$ , then arm  $k$  is (generically) chosen only finitely often. If the infimum of the rewards are equal, then both arms are played infinitely often.

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## 1. INTRODUCTION

There has been a flurry of interest in studying the asymptotic dynamics of Bayesian learning and control in economic environments. In one set of papers including Easley and Kiefer (1988), Easley and Kiefer (1989), Kiefer and Nyarko (1988), Kiefer and Nyarko (1989), McLennan (1987), Feldman and McLennan (1989), and Bikhchandani and Sharma (1990), the authors analyze single agent decision problems in which there is a tradeoff between current period expected reward and the expected value of the information generated by the current period action. In another strand of the literature, Blume and Easley (1984), Bray and Kreps (1987), Feldman (1987a), and Feldman (1987b) focus on properties of the tail of the sequence of beliefs and outcomes for economies with many passively learning agents. Specifically, these latter papers consider whether Bayesian learning by agents with a correct specification of the underlying structure but uncertainty regarding the parameter values is a sufficient condition to assure convergence to a stationary rational expectations equilibrium.

These articles, as well as important earlier contributions of Cyert and DeGroot (1973), Rothschild (1974), and Townsend (1978), have the following common framework. From the vantage point of the economic actors, the set of possible complete descriptions of the relevant time-invariant economic data can be represented as a separable metric space  $\Theta$  with Borel  $\sigma$ -field  $B(\Theta)$ . The actors in the model, uncertain as to the "true"  $\theta_0 \in \Theta$ , have prior beliefs  $\mu$  on  $(\Theta, B(\Theta))$ <sup>1</sup>, and an induced probability  $P_\mu$  on an infinite horizon outcome space. Denote by  $\{\mu_t\}$  the sequence of posterior beliefs. A result common to this literature is a "theorem" that with probability one  $\mu_t \Rightarrow \mu_\infty$ , where  $\mu_\infty$  is the posterior probability conditioned on the limit sub- $\sigma$ -field.

In most of the recent papers (exceptions are Feldman and McLennan (1989) and Bikhchandani and Sharma (1990)), this a.s. convergence is established by using the fact that the sequence of posterior beliefs are a martingale with respect to the probability  $P_\mu$ . It follows from the Martingale Convergence Theorem that with  $P_\mu$  probability one, the Bayesian beliefs converge to some (possibly random) limit belief. In contrast, consider the distribution of outcomes and beliefs with respect to the probability measure  $P_{\theta_0}$ , the probability induced by the "true" parameter  $\theta_0$ . Intuitively,  $P_{\theta_0}$  is the belief of a passive observer who attaches probability one to  $\theta_0$  being the truth. One might also inquire as to whether convergence of Bayesian beliefs is obtained with respect to the measure  $P_{\theta_0}$ . A major point of this paper is to stress that without additional conditions, the answer is negative.

To elaborate on this distinction, suppose that in period  $t = 0, 1, \dots$ , agents observe outcomes in a separable metric space  $Y$ . Given the behavioral rules of the agents, each parameter value  $\theta \in \Theta$  induces a probability measure  $P_\theta$  on the product space  $Y^\infty$ . The prior  $\mu$  induces a measure  $P_\mu$  on  $Y^\infty$  defined by  $P_\mu(A) = \int P_\theta(A) P_\mu(d\theta)$ . The application of the Martingale Convergence Theorem yields a.s. convergence of  $\{\mu_t\}$ , where the a.s. statement is with respect to the probability measure  $P_\mu$ . But this does not imply that for any particular  $\theta$  that  $\mu_t \Rightarrow \mu_\infty$  with  $P_\theta$  probability one, even if  $\theta$  is in the support of  $\mu$ .

One might hope to establish a result that for a "large" class of priors, posteriors converge for a "large" class of parameter values. When  $\Theta$  can be embedded in finite-dimensional Euclidean space, one has recourse to Lebesgue measure  $m$  (restricted to  $\Theta$ ) as a natural notion of size. Then if  $m \ll \mu$  the exceptional  $\theta$  set  $\{\theta: P_\theta(\{\mu_t \neq \mu\}) < 1\}$ ,

has measure zero. But in many naturally occurring settings  $\Theta$  is not finite dimensional, and since there is no infinite dimensional analogue of Lebesgue measure, a measure-theoretic criterion is unavailable.

In lieu of a reference measure to evaluate size, the customary procedure is to resort to the topological notion of category. Residual subsets are deemed to be large or *generic*, and subsets of first category (which are complements of residual subsets) are regarded as small. Freedman (1965) proved that when outcomes are I.I.D. taking values in a countable set, that for a residual set of parameter values and priors, posterior beliefs do not converge. In Section 3 of this paper we extend Freedman's result to outcomes in non-finite, complete, separable metric spaces.

Using the results of Section 3, in Sections 4 and 5 of the paper we analyze a two-armed bandit problem with geometric discounting where arm  $k$  yields an outcome in a complete, separable metric space  $Y_k$ . If the infimum of the possible rewards from playing arm  $k$  is less than the infimum from playing arm  $k'$ , then for a residual family of parameter values and priors, arm  $k$  is with  $P_\theta$  probability one chosen only finitely often. If the infimum of the rewards are equal, then both arms are played infinitely often.

## 2. NOTATION AND MATHEMATICAL PRELIMINARIES

### 2.1. Notational Conventions and Definitions

The set of real numbers is denoted by  $\mathbf{R}$ . If  $X$  is a topological space, then the Borel  $\sigma$ -field is denoted by  $B(X)$ . The set of probability measures on  $(X, B(X))$  is denoted by  $P(X)$ . For  $x \in X$ , the Dirac measure  $\delta_x \in P(X)$  is defined by  $\delta_x(A) = 1$  if  $x \in A$ .

If  $(X, d)$  is a metric space,  $f: X \rightarrow \mathbf{R}$  is a Lipschitz function if for some  $K < \infty$ ,  $\sup_{x \neq y} \{|f(x) - f(y)|/d(x, y)\} < K$ . If  $f$  is Lipschitz, the Lipschitz seminorm  $\|f\|_L$  is defined  $\|f\|_L = \sup_{x \neq y} \{|f(x) - f(y)|/d(x, y)\}$ . If  $f$  is a bounded Lipschitz function, the *bounded Lipschitz* norm is  $\|f\|_{BL} = \|f\|_L + \|f\|_\infty$  where  $\|f\|_\infty$  denotes the usual sup norm. The set of all real-valued, bounded Lipschitz functions on  $(X, d)$  is denoted by  $BL(X, d)$ . Endowed with the bounded Lipschitz norm,  $BL(X, d)$  is a Banach space (see e.g. Dudley (1989, Section 11.2)).

The *dual bounded Lipschitz* or *Dudley metric*  $\beta$  on  $P(X)$  is defined by

$$\beta(P, Q) = \sup\{|\int f dP - \int f dQ|: \|f\|_{BL} \leq 1\},$$

for  $P, Q \in P(X)$ . If  $X$  is separable,  $\beta$  metrizes the topology of weak convergence on  $P(X)$ . Further details on the properties of  $\beta$  can be found in Dudley (1966) and Dudley (1989).

### 2.2. A Brief Review of Baire Category Theory

For ease of reference, we summarize some needed facts pertaining to Baire category. Standard references include Kelley (1985, pp. 200-203), Oxtoby (1980) and Royden (1988, Section 7.8). Let  $X$  be a metric space. A set  $E \subset X$  is *nowhere dense* if  $\bar{E}$  has empty interior. A set  $E$  is of *first category* or *meager* if it is the union of a countable collection of nowhere dense sets. If a set is not of first category then it is of *second category*. The complement of a set of first category is a *residual set*.

According to the Theorem of Baire Royden (1988, Theorem 7.27), if  $X$  is a complete metric space then the intersection of a countable family of open dense subsets of  $X$  is itself a dense subset of  $X$ .

### 3. GENERIC NONCONVERGENCE OF POSTERIOR WITH I.I.D. OUTCOMES IN COMPLETE, SEPARABLE METRIC SPACES

#### 3.1. Assumptions and Results

We first describe an index set  $\Lambda$  and a sequence  $Z_1, Z_2, \dots$  of i.i.d. random variables defined on a probability space  $(\Sigma, \mathcal{S}, P_\lambda)$  where  $\lambda \in \Lambda$ . The natural interpretation will be that the outcomes are sequentially observed by a Bayesian statistician for whom the “true”  $\lambda$  is initially unknown and has prior belief  $\mu$ . Building upon the work of Freedman (1965), we will investigate the topological size of the set of pairs  $(\lambda, \mu)$  for which the sequence of Bayesian posterior beliefs converges  $P_\lambda$  a.s. or in  $P_\lambda$  probability to some limit posterior belief.

The sequence  $\{Z_n\}$  takes values in  $Z$ , a non-finite, complete, separable metric space with Borel  $\sigma$ -field  $B(Z)$ . The probability distribution of  $Z_n$  is an element of  $\Lambda = \{\lambda \in P(Z) : \lambda \ll \nu\}$ , where  $\nu$  is a  $\sigma$ -finite measure on  $(Z, B(Z))$  with non-finite support. Without loss of generality we work in representation space and so define  $\Sigma = Z^\infty$ ,  $\mathcal{S} = B(Z) \times B(Z) \times \dots$ , and  $P_\lambda = \lambda \times \lambda \times \dots$ . The function  $Z_n : \Sigma \rightarrow Z$  is the projection of  $\Sigma$  onto  $Z$ , defined by  $Z_n(z_1, z_2, \dots) = z_n$ .

To address the question of convergence of posterior beliefs we need topologies on  $\Lambda$  and  $P(\Lambda)$ . We will make use of two topologies on  $\Lambda$ , the total variation topology  $\mathcal{T}_1$  and the topology of weak convergence  $\mathcal{T}_w$ .  $\mathcal{T}_1$  is induced by the  $L^1$  metric  $d_1$  defined by  $d_1(\lambda, \lambda') = \int_Z \left| \frac{d\lambda}{d\nu} - \frac{d\lambda'}{d\nu} \right| d\nu$ . An essential fact is that  $(\Lambda, d_1)$  is a complete, separable

metric space. (Completeness follows from the completeness of  $L^1(Z, B(Z), \nu)$ , and for separability see e.g., Strasser (1985, Lemma 4.1).) In contrast, defining  $\beta_\Lambda$  as the Dudley metric on  $\Lambda$  (which generates the topology  $\mathcal{T}_w$ ), the metric space  $(\Lambda, \beta_\Lambda)$  is separable, but not complete. Conveniently, the Borel  $\sigma$ -field of  $(\Lambda, \mathcal{T}_1)$  is the same as the Borel  $\sigma$ -field of  $(\Lambda, \mathcal{T}_w)$  (see Strasser (1985, Theorem 4.7). So without ambiguity we can denote the Borel sets of  $\Lambda$  by  $B(\Lambda)$ .

A *prior distribution* is a probability measure  $\mu$  on  $(\Lambda, B(\Lambda))$ . As indicated above, informally one can imagine that there is a Bayesian statistician who may not know the “true”  $\lambda$ , but has a prior  $\mu$ . Since  $\Lambda$  has two topologies, there are two weak topologies on  $P(\Lambda)$  denoted in the obvious way by  $\mathcal{H}_1$  and  $\mathcal{H}_w$ , with  $\mathcal{H}_w$  weaker than  $\mathcal{H}_1$ .  $\mathcal{H}_w$  and  $\mathcal{H}_1$  generate the same  $\sigma$ -field, which we denote by  $B(P(\Lambda))$ . Convergence with respect to the  $\mathcal{H}_1$  topology is denoted by  $\Rightarrow$ . Convergence with respect to the  $\mathcal{H}_w$  topology is denoted by  $\overset{w}{\Rightarrow}$ . In this section of the paper the symbol  $\beta$  denotes the Dudley metric on  $P(\Lambda)$  with respect to the  $d_1$  metric on  $\Lambda$ . It follows from Billingsley (1968, p. 239) and Dudley (1989, Corollary 11.5.5) that the metric space  $(P(\Lambda), \beta)$  is complete and separable with  $\beta$  generating the topology  $\mathcal{H}_1$ .

The *updating rule*  $\Gamma : P(\Lambda) \times Z \rightarrow P(\Lambda)$  is a measurable function with the property that for each  $\mu \in P(\Lambda)$ ,  $\Gamma(\mu, \cdot)$  is a regular version of conditional probability with respect to the prior probability  $\mu$ . The existence of such a



function is established by Dynkin and Yushkevich (1979, p. 263). The  $n$ -period updating rule is  $\Gamma_n: P(\Lambda) \times \Sigma \rightarrow P(\Lambda)$ , recursively defined by  $\Gamma_1(\mu, \sigma) = \Gamma(\mu, Z_1(\sigma))$  and  $\Gamma_n(\mu, \sigma) = \Gamma(\Gamma_{n-1}(\mu, \sigma), Z_n(\sigma))$  for  $n \geq 2$ .

A pair  $(\lambda, \mu) \in \Lambda \times P(\Lambda)$  is  $\mathcal{H}_1$ -consistent if  $P_\lambda(\{\sigma: \Gamma_n(\mu, \sigma) \Rightarrow \delta_\lambda\}) = 1$ . A pair  $(\lambda, \mu) \in \Lambda \times P(\Lambda)$  is  $\mathcal{H}_w$ -consistent if  $P_\lambda(\{\sigma: \Gamma_n(\mu, \sigma) \xrightarrow{w} \delta_\lambda\}) = 1$ . (When  $v$  has countable support,  $\mathcal{H}_1 = \mathcal{H}_w$  and so the two definitions of consistency are identical.) Freedman (1963) proved that when  $Z$  is finite and  $\mu$  has full support, that  $(\lambda, \mu)$  is consistent for all  $\lambda \in P(\Lambda)$ . It would be natural to conjecture a similar result for when  $Z$  is not finite. Indeed, the consistency result for the finite outcome case has been generalized in a well-known paper of Schwartz (1965) and more recently by Barron (1988). However, Freedman (1965) demonstrated that in a topological sense “most” pairs are not consistent (in either sense) when  $v$  has countable support, even if it is required that the prior  $\mu$  has full support. More precisely, defining  $S = \{\mu \in P(\Lambda): \text{supp } \mu = \Lambda\}$ , Freedman proved the striking result that there exist sets  $R_\Lambda \subset \Lambda$  and  $R_{P(\Lambda)} \subset S$ , residual in  $\Lambda$  and  $P(\Lambda)$  respectively (which implies that  $R_\Lambda \times R_{P(\Lambda)}$  is residual in  $\Lambda \times P(\Lambda)$ ), such that for  $(\lambda, \mu) \in R_\Lambda \times R_{P(\Lambda)}$ :

$$\limsup_{n \rightarrow \infty} \int_\Sigma \Gamma_n(\mu, \sigma)(G) P_\lambda(d\sigma) = 1, \text{ for all nonempty open subsets } G \subset \Lambda. \text{ A corollary is that for } (\lambda, \mu) \text{ in the}$$

residual set  $R_\Lambda \times R_{P(\Lambda)}$ ,  $P_\lambda(\{\sigma: \Gamma_n(\mu, \sigma) \Rightarrow \delta_\lambda\}) = 0$ .

In the next subsection we show that these non-convergence results of Freedman (1965) extend to the case where  $v$  is any  $\sigma$ -finite measure with non-finite support. Endowing  $\Lambda$  with the  $d_1$  metric, and defining  $\mathcal{R} = \{(\lambda, \mu) \in \Lambda \times P(\Lambda): \limsup_{n \rightarrow \infty} \int_\Sigma \Gamma_n(\mu, \sigma)(G) P_\lambda(d\sigma) = 1 \text{ for all open } G \subset \Lambda\}$ , we prove the following theorem and corollary.

**THEOREM 3.10.**  $\mathcal{R}$  is a residual subset of  $\Lambda \times P(\Lambda)$ . Furthermore, defining  $S = \{\mu \in P(\Lambda): \text{supp } \mu = \Lambda\}$  and  $\mathcal{R}_S = \mathcal{R} \cap (\Lambda \times S)$ ,  $\mathcal{R}_S$  is a residual subset of  $\Lambda \times P(\Lambda)$ .

**COROLLARY 3.11.** All  $(\lambda, \mu) \in \mathcal{R}$  are neither  $\mathcal{H}_1$  or  $\mathcal{H}_w$ -consistent.

### 3.2. Proof of Theorem and Corollary

While the basic structure of the proof is closely resembles the proof of Freedman (1965) of his Theorem, some modification and extension is required to adapt the argument to cover a non-discrete outcome space. In particular, Proposition 3.5 requires a different method of proof than the comparable intermediate result in Freedman (1965).

We start with some definitions. Unless otherwise indicated, where relevant it should be understood that  $\Lambda$  is endowed with the  $\mathcal{H}_1$  topology. Define  $h: \Lambda \times Z \rightarrow \mathbb{R}$  as  $(B(\Lambda) \times B(Z))$  measurable function such that  $h(\lambda, \cdot)$  is a density for  $\lambda$  with respect to  $v$  (a proof of existence of such a function is provided by Strasser (1985, Lemma 4.6)). Let  $\Lambda_+ = \{\lambda \in \Lambda: h(\lambda, z) > 0, v \text{ a.e.}\}$  and define  $\Lambda_0 = \sim \Lambda_+$ . The set of probability measures on  $(\Lambda, B(\Lambda))$  that assign strictly positive probability to  $\Lambda_+$  is  $P_+(\Lambda) = \{\mu \in P(\Lambda): \mu(\Lambda_+) > 0\}$ .

$\Lambda_+$  and  $P_+(\Lambda)$  are topologically “large” in the sense that each is a residual subset of a complete, separable metric space. In contrast,  $\Lambda_0$  and  $P(\Lambda_0) = \{\mu \in P(\Lambda): \mu(\Lambda_0) = 1\}$  are of first category, albeit dense in respectively  $\Lambda$  and  $P(\Lambda)$ .

LEMMA 3.1.  $\Lambda_+ \subset \Lambda$  and  $P_+(\Lambda) \subset P(\Lambda)$  are respectively dense  $G_\delta$  (and hence residual) subsets of  $\Lambda$  and  $P_+(\Lambda)$ .  $\Lambda_0 \subset \Lambda$  and  $P(\Lambda_0) \subset P(\Lambda)$  are dense sets of first category.

*Proof.* (i) We first establish the properties of  $\Lambda_+$  and  $P(\Lambda_+)$ . Define  $\Lambda^j = \{\lambda \in \Lambda: v(\{z: h(\lambda, z) = 0\}) < j^{-1}\}$  for  $j = 1, 2, \dots$ .  $\Lambda^j$  is an open, dense subset of  $\Lambda$ . So by the Baire Category Theorem (see e.g., Royden (1988, Theorem 7.27))  $\Lambda_+ = \bigcap_{j=1}^{\infty} \Lambda^j$  is a dense  $G_\delta$  set and hence residual (Royden (1988, Theorem 7.30)).

The claim that  $P_+(\Lambda)$  is a dense  $G_\delta$ , follows from Theorem 3.15 of Dubins and Freedman (1964). (Dubins and Freedman have a compactness assumption in Section 3 of their paper; but inspection of their proof reveals that completeness and separability is a sufficient condition.)

(ii) We now verify the properties of  $\Lambda_0$  and  $P(\Lambda_0)$ . Since  $\Lambda_0 = \sim \Lambda_+$  and  $P(\Lambda_0) = \sim P_+(\Lambda)$ , and  $\Lambda_+$  and  $P_+(\Lambda)$  are residual, by definition  $\Lambda_0$  and  $P(\Lambda_0)$  are of first category. To prove denseness, for arbitrary  $\lambda \in \Lambda$  choose a sequence  $\lambda^k \rightarrow \lambda$  such that  $h(\lambda^k, z) = 0$  on a set of positive  $v$  measure and  $h(\lambda^k, \cdot)$  converges in  $v$  measure to  $h(\lambda, \cdot)$ . (The existence of such a sequence follows from the fact that for every  $\alpha \in [0, 1]$  there exists  $A_\alpha \in \mathcal{B}(Z)$  such that  $\alpha = \int h(\lambda, z) v(dz)$ .) But then  $\lambda^k \rightarrow \lambda$ , establishing the denseness of  $\Lambda_0$ . The density of  $P(\Lambda_0)$  now follows from

Theorem II.6.3 of Parthasarathy (1967). ■

Let  $D = \{\alpha_1, \alpha_2, \dots\}$  be a countable, dense subset of  $\Lambda_+$  and hence dense subset of  $\Lambda$ . We now construct a sequence  $M_1, M_2, \dots$  with  $M_k \subset P(\Lambda)$  such that for all  $\lambda \in \Lambda_+$  and  $\mu \in M_k$ ,  $\Gamma_n(\mu, \sigma)$  converges  $P_\lambda$  a.s. to  $\delta_{\alpha_k}$ , the Dirac measure on  $\alpha_k$ . Proceeding, we define  $M_k \subset P(\Lambda)$  by  $M_k = \{\mu \in P(\Lambda): (i) \mu \text{ has finite support, (ii) } \mu(\{\alpha_k\}) > 0, \text{ and (iii) } \mu(\Lambda_0) = 1 - \mu(\{\alpha_k\})\}$ . The set  $M \subset P(\Lambda)$  is defined by  $M = \bigcup_{k=1}^{\infty} M_k$ .

LEMMA 3.2. For  $k = 1, 2, \dots$ ,  $M_k$  is a dense subset of  $P(\Lambda)$ .

*Proof.* Select  $\alpha_k \in D$ ,  $\mu \in P(\Lambda)$  and define  $\Xi = \{\gamma \in P(\Lambda): \text{supp } \gamma \text{ is finite, and } \gamma(\Lambda_0) = 1\}$ . Since  $\Lambda_0$  is dense in  $\Lambda$  (Lemma 2.1), by Theorem II.6.3 of Parthasarathy (1967),  $\Xi$  is dense in  $P(\Lambda)$ . So there exists  $\gamma^m \Rightarrow \mu$ , with  $\gamma^m \in \Xi$ . Define  $\mu^m = m^{-1} \cdot \delta_{\alpha_k} + (1 - m^{-1}) \cdot \gamma^m$ . Since  $\mu^m \in M_k$  and  $\mu^m \Rightarrow \mu$ , the proof is complete. ■

Given the above definitions, it is intuitive that if the prior  $\mu \in M_k$  and  $\lambda_0 \in \Lambda_+$ , then with  $P_{\lambda_0}$  probability one, any a priori alternative to  $\lambda_0$  will eventually be deemed impossible and the posterior belief will converge to  $\delta_{\alpha_k}$ .

LEMMA 3.3. For  $\lambda_0 \in \Lambda_+$  and  $\mu \in M_k$ ,  $P_{\lambda_0}(\{\sigma \in \Sigma: \Gamma_n(\mu, \sigma) \Rightarrow \delta_{\alpha_k}\}) = 1$ .

*Proof.* Suppose  $\text{supp } \mu = \{\alpha_k, \lambda_1, \lambda_2, \dots, \lambda_J\}$  where  $\lambda_j \in \Lambda_0$  for  $j = 1, \dots, J$ . Let  $A_j = \{z \in Z: h(\lambda_j, z) = 0\}$ . Define the exceptional set  $E_j = \{\sigma = (z_1, z_2, \dots) \in \Sigma: z_k \notin A_j \text{ for } k = 1, 2, \dots\}$ .  $P_{\lambda_0}(E_j) = 0$ , and for  $\sigma \notin E_j$ ,  $\Gamma_n(\mu, \sigma)(\{\lambda_j\}) > 0$  only finitely often. Since  $\{\lambda_1, \dots, \lambda_J\}$  is finite,  $\Gamma_n(\mu, \sigma)(\{\alpha_k\}) < 1$  only finitely often. ■

Now for  $k = 1, 2, \dots$ , let  $\{O_{km}\}_{m=1}^{\infty}$  be a decreasing sequence of open subsets of  $\Lambda$  with  $\bar{O}_{k,m+1} \subset O_{km}$  and  $\bigcap_{m=1}^{\infty} O_{km} = \{\alpha_k\}$ . Let  $g_{km}: \Lambda \rightarrow \mathbf{R}$  be a bounded Lipschitz function such that  $\|g_{km}\|_{BL} \leq 1$ ,  $g_{km}$  equals one on  $O_{k,m+1}$  and vanishes on  $\sim O_{km}$ . The existence of functions satisfying these conditions follows from Proposition 11.2.3 of Dudley (1989).

LEMMA 3.4. *If  $\lambda_0 \in \Lambda_+$  and  $\mu \in M_k$ , then  $\lim_{n \rightarrow \infty} \int_{\Omega} [\int_{\Lambda} g_{km}(\lambda) \Gamma_n(\mu, \sigma)(d\lambda)] P_{\lambda_0}(d\sigma) = 1$ .*

*Proof.* Define  $G_{kmn}(\sigma) = \int_{\Lambda} g_{km}(\lambda) \Gamma_n(\mu, \sigma)(d\lambda)$ . Applying Lemma 3.3 and the definition of weak convergence,

$\lim_{n \rightarrow \infty} G_{kmn}(\sigma) = 1, P_{\lambda_0}$  a.s.. So by the Dominated Convergence Theorem,  $\lim_{n \rightarrow \infty} \int_{\Sigma} G_{kmn}(\sigma) P_{\lambda_0}(d\sigma) = 1$ . ■

For arbitrary prior beliefs  $\mu \in P(\Lambda)$  and ‘true’ parameter  $\lambda \in \Lambda$ , the probability law (with respect to the measure  $P_{\lambda}$ ) of the posterior mapping  $\Gamma_n(\mu, \cdot): \Sigma \rightarrow P(\Lambda)$  may vary with the choice of versions of conditional probability. It is easily confirmed, however, that if  $\mu \in P_+(\Lambda)$  then with  $P_{\lambda}$  probability one, any two versions of conditional probability will agree. The next task is to verify that if we restrict attention to prior beliefs  $\mu \in P_+(\Lambda)$ , then from the perspective of statistical observer who “knows”  $\lambda$ , the expected value of the Bayesian’s posterior expectation of  $g_{km}$  is a continuous function of the prior  $\mu$  and the true parameter  $\lambda$ . This is a lengthy exercise with the details provided in the Appendix. Since  $P_+(\Lambda)$  is a residual set, for the purposes of this paper this restricted continuity result suffices.

PROPOSITION 3.5. *The function  $\Phi_{kmn}: \Lambda \times P_+(\Lambda) \rightarrow \mathbf{R}$  defined by*

$$\Phi_{kmn}(\lambda, \mu) = \int_{\Sigma} \int_{\Lambda} g_{km}(\lambda') \Gamma_n(\mu, \sigma)(d\lambda') P_{\lambda}(d\sigma), \text{ is continuous for all } k, m \text{ and } n.$$

*Proof.* See Appendix. ■

The remaining steps needed for the proof of Theorem 3.11 mimic Freedman (1965). To make the paper self-contained, modulo notational changes (and filling in some details) we replicate Freedman’s ingenious argument.

Define for  $k, j, m, n = 1, 2, \dots$ , the set  $R_{kmjn} \subset \Lambda \times P_+(\Lambda)$  by:

$$R_{kmjn} = \{(\lambda, \mu) \in \Lambda \times P_+(\Lambda): \int_{\Sigma} [\int_{\Lambda} g_{km}(\lambda') \Gamma^n(\mu, \sigma)(d\lambda')] P_{\lambda}(d\sigma) \leq 1 - j^{-1}\}.$$

And define  $\mathcal{S} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} R_{kmjn}$ . The set of  $(\lambda, \mu)$  pairs such that with  $P_{\lambda}$  probability one, the Bayesian’s posterior belief essentially concentrates in every open  $\Lambda$  set infinitely often is:

$$\mathcal{B} = \{(\lambda, \mu) \in \Lambda \times P(\Lambda): \limsup_{n \rightarrow \infty} \int_{\Sigma} \Gamma_n(\mu, \sigma)(G) P_{\lambda}(d\sigma) = 1, \forall \text{ open } G \subset \Lambda\}.$$

To aid the reader, we outline the structure of the remainder of the proof. In Proposition 3.6 and 3.7 we establish that: (i)  $\mathcal{S}$  is of first category in  $\Lambda \times P_+(\Lambda)$ , and (ii) that  $[\Lambda \times P_+(\Lambda)] \setminus \mathcal{B} \subset \mathcal{S}$  implying that  $\mathcal{B}$  is

residual in  $[\Lambda \times P_+(\Lambda)]$ . In conjunction with the fact that a residual subset of a residual subspace is residual (a consequence of Lemma 3.9), this establishes that  $\mathcal{R}$  is residual in  $\Lambda \times P(\Lambda)$ .

**PROPOSITION 3.6.** *For all  $k, m, j \geq 1$ ,  $\cap_{n=i}^{\infty} R_{kmjn}$  is a relatively closed, nowhere dense subset of  $\Lambda \times P_+(\Lambda)$ .  $\mathcal{V}$  is of first category in  $\Lambda \times P_+(\Lambda)$ .*

*Proof.* By Proposition 3.5,  $R_{kmjn}$  is closed in  $\Lambda \times P_+(\Lambda)$  and so  $\cap_{n=i}^{\infty} R_{kmjn}$  is closed in  $\Lambda \times P_+(\Lambda)$ . By Lemma 3.4, if  $(\lambda, \mu) \in \Lambda_+ \times M_k$  then  $(\lambda, \mu) \notin \cap_{n=i}^{\infty} R_{kmjn}$ . By Lemma 3.2  $\Lambda_+ \times M_k$  is dense in  $\Lambda \times P_+(\Lambda)$ , and so  $\cap_{n=i}^{\infty} R_{kmjn}$  is nowhere dense in  $\Lambda \times P_+(\Lambda)$ . Since  $\cap_{n=i}^{\infty} R_{kmjn}$  is a closed, nowhere dense subset of  $\Lambda \times P_+(\Lambda)$ ,  $\mathcal{V}$  is a countable union of closed, nowhere dense sets, and so is of first category in  $\Lambda \times P_+(\Lambda)$ . ■

**PROPOSITION 3.7.**  $[\Lambda \times P_+(\Lambda)] \setminus \mathcal{R} \subset \mathcal{V}$

*Proof.* Suppose  $(\lambda, \mu) \in [\Lambda \times P_+(\Lambda)] \setminus \mathcal{R}$ . Then there exists  $\varepsilon > 0$  and an open set  $G \subset \Lambda$ , such that  $\limsup_n \int_{\Omega} \Gamma_n(\mu, \sigma)(G) P_{\lambda}(d\sigma) < 1 - \varepsilon$ . Choose  $k$  and  $m$  so that  $\{\lambda': g_{km}(\lambda') > 0\} \subset G$ . Choose  $j$  so that  $1 - j^{-1} > 1 - \varepsilon$  and choose  $n_0$  so that for all  $n \geq n_0$ ,  $\int_{\Omega} \Gamma_n(\mu, \sigma)(G) P_{\lambda}(d\sigma) < 1 - j^{-1}$ . Then  $(\lambda, \mu) \in R_{kmjn}$  and so  $(\lambda, \mu) \in \cap_{n=n_0}^{\infty} R_{kmjn}$ , which by the definition of  $\mathcal{V}$  implies that  $(\lambda, \mu) \in \mathcal{V}$ . ■

**COROLLARY 3.8.**  $\mathcal{R} \cap [\Lambda \times P_+(\Lambda)]$  is a residual subset of  $\Lambda \times P_+(\Lambda)$  in the relative topology.

*Proof.* By Propositions 3.6 and 3.7,  $[\Lambda \times P_+(\Lambda)] \setminus \mathcal{R}$  is of first category in  $\Lambda \times P_+(\Lambda)$ , so the relative complement  $R \cap [\Lambda \times P_+(\Lambda)]$  is residual in  $\Lambda \times P_+(\Lambda)$ . ■

**LEMMA 3.9.** *Suppose  $(Y, T)$  is a topological space and  $(Y_0, U)$  is a subspace where  $U$  is the relative topology.*

- (i) *If  $A \subset Y_0$  is nowhere dense in  $(Y_0, U)$  then  $A$  is nowhere dense in  $(Y, T)$ .*
- (ii) *If  $Y_0$  is a residual subset of  $Y$  and  $B \subset Y_0$  is residual in  $(Y_0, U)$  then  $B$  is a residual subset of  $Y$ .*

*Proof.* (i) Suppose  $A$  is not nowhere dense in  $Y$ . Then there is an open set  $G \subset Y$  with  $G \subset \bar{A}$ . Furthermore,  $G \cap Y_0 \subset \bar{A} \cap Y_0 = \text{cl}_{Y_0} A$ , the closure of  $A$  in  $(Y_0, U)$ , where the equality follows from Theorem 1.16 of Kelley (1985). But this contradicts  $A$  being nowhere dense in  $Y_0$ .

(ii) To establish that  $B$  is residual in  $Y$ , observe that there exists a sequence of sets  $\{A_i\}$  with  $A_i \subset Y_0$ ,  $Y_0 \setminus B = \cup_i A_i$ , and  $A_i$  nowhere dense in  $(Y_0, U)$ . So by (i),  $A_i$  is nowhere dense in  $Y$ , implying that  $Y_0 \setminus B$  is of first category in  $Y$ , so  $Y_0 \setminus B \cup Y \setminus Y_0$  is of first category in  $Y$ , and  $B = \sim(Y_0 \setminus B \cup Y \setminus Y_0)$  is residual in  $Y$ . ■

*Proof of THEOREM 3.10.* By Lemma 3.1,  $P_+(\Lambda)$  is a residual subset of  $P(\Lambda)$ . So by Theorem 15.3 of Oxtoby (1980),  $\Lambda \times P_+(\Lambda)$  is a residual subset of  $\Lambda \times P(\Lambda)$ . By Corollary 3.8,  $\mathcal{R} \cap [\Lambda \times P_+(\Lambda)]$  is a residual subset of  $\Lambda \times P_+(\Lambda)$ , and so by Lemma 3.9,  $\mathcal{R} \cap [\Lambda \times P_+(\Lambda)]$  is a residual subset of  $\Lambda \times P(\Lambda)$ . This completes the proof that

$\mathcal{R}$  is a residual subset of  $\Lambda \times P(\Lambda)$ . Applying Theorem 3.13 of Dubins and Freedman (1964),  $S$  is a residual subset of  $P(\Lambda)$ , and by Oxtoby (1980, Theorem 15.3)  $\Lambda \times S$  is a residual subset of  $\Lambda \times P(\Lambda)$ . So  $\mathcal{R} \cap (\Lambda \times S) = \mathcal{R}_S$  is residual. ■

*Proof of COROLLARY 3.11.* Pick  $(\lambda, \mu) \in \mathcal{R}$ ,  $\varepsilon > 0$ , and choose a set  $G \subset \Lambda$  open in the  $\mathcal{T}_w$  topology with  $\mathcal{T}_w$ -closure  $\bar{G}$  such that  $\lambda \notin \bar{G}$ . Since  $(\lambda, \mu) \in \mathcal{R}$ ,  $P_\lambda$  a.s.  $\Gamma_n(\mu, \sigma)(G) > 1 - \varepsilon$  infinitely often,  $\Gamma_n(\mu, \sigma)(\bar{G}) < \varepsilon$  i.o.. But by the standard characterization of weak convergence (see e.g., Billingsley (1968, Theorem 2.1)) this implies that  $\Gamma_n(\mu, \sigma)$  does not converge  $\xrightarrow{w}$  to  $\delta_\lambda$ . And since  $\mathcal{T}_w$  is weaker than  $\mathcal{T}_1$ ,  $\Gamma_n(\mu, \sigma)$  does not converge  $\Rightarrow$  to  $\delta_\lambda$ . ■

#### 4. AN APPLICATION: INFINITE HORIZON BANDIT PROBLEMS WITH DISCOUNTING

##### 4.1. Introduction

In this section we model a Bayesian decision-maker who faces an infinite-horizon two-armed bandit problem and geometrically discounts future rewards. In a well-known article Rothschild (1974) applied the bandit framework to model the decision-making of a monopolist who could charge one of two prices and was uncertain of the distribution of demand associated with each price. An extended discussion of economic applications of bandit problems is provided by Kiefer (1989)

To orient the reader we first provide an informal description of the Bayesian's optimization problem. We then formally define the relevant probability spaces and reformulate the decision-problem as a dynamic programming problem. Using the Gittens Index, we provide a simple characterization the behavioral rules of the decision-maker. In Section 5, we apply Theorem 3.10 to describe the asymptotic behavior of the decision-maker.

Time periods are indexed by  $t = 0, 1, 2, \dots$ . In period  $t$  the decision-maker selects an *action* or *bandit-arm*  $x(t) \in X = \{x_1, x_2\}$ . After choosing action  $x(t) = x_k$ , the realization  $y(t) \in Y_k$  of a random element  $Y(t)$  is observed, and a period reward  $r_k(y(t))$  is received. Conditional upon the Bayesian choosing  $x(t) = x_k$ , the probability distribution of  $Y(t)$  is an element of  $\theta_k \in \Theta_k \subset P(Y_k)$ . Defining  $Y = Y_1 \cup Y_2$ , and  $r: X \times Y \rightarrow \mathbf{R}$  by  $r(x_k, y) = r_k(y)$ , the total reward or utility from the stream  $(x(0), y(0), x(1), y(1), x(2), \dots)$  is  $\sum_{t=0}^{\infty} \alpha^t \cdot r(x(t), y(t))$ , where the discount factor  $\alpha \in [0, 1)$ .

The decision-maker may initially be uncertain of the "true"  $\theta_1$  and  $\theta_2$ . Defining  $\Theta = \Theta_1 \times \Theta_2$ , her initial beliefs are given by a prior probability  $\mu \in P(\Theta)$ . Since the decision-maker's choice of action at time  $t$  may be influenced by previously observed random outcomes, the action  $x(t)$  is the realization of a random variable  $X(t)$ . A *policy* is a sequence of random variables  $\{X(t)\}$  taking values in  $X$ , that are measurable with respect to the information (i.e., sub- $\sigma$ -fields) generated by past outcomes. The objective of the decision-maker is to select a policy that maximizes  $E_\mu[\sum_{t=1}^{\infty} \alpha^{t-1} \cdot r(X(t), Y(t))]$ , where the symbol  $E_\mu$  (informally) denotes expectation with respect to a probability measure on the underlying probability space consistent with the prior  $\mu$ .



#### 4.2. Formal Specification of the Probability Spaces

The set  $Y_k$  of outcomes that might arise from selecting action  $x_k$  is a non-finite, complete, separable metric space with Borel  $\sigma$ -field  $B(Y_k)$ . Define  $\Omega_k = Y_k \times Y_k \times \dots$  with  $F_k$  the corresponding product  $\sigma$ -field. We write  $F_{k0}$  for the  $n$ -fold product  $\sigma$ -field  $B(Y_k) \times B(Y_k) \times \dots \times B(Y_k)$  with  $F_{k0} = \{\Omega_k, \emptyset\}$ . The measurable space  $(\Omega, F)$  is defined by  $\Omega = \Omega_1 \times \Omega_2$ , and  $F = F_1 \times F_2$ . The projection function  $\pi_k: \Omega \rightarrow Y_k$  is defined by  $\pi_k(j, \omega_{k1}, \omega_{k2}, \dots) = \omega_{kj}$ . The interpretation of  $\Omega_k$  is that for  $\omega_k \in \Omega_k$ ,  $\pi_k(j, \omega_k)$  is the outcome on the  $j$ 'th occasion that action  $x_k$  is selected. Slightly abusing notation, when convenient we treat  $\pi_k(j, \cdot)$  as a function from  $\Omega$  to  $Y$  defined by  $\pi_k(j, \omega_1, \omega_2) = \pi_k(j, \omega_{kj})$ .

The set of a priori possible probability laws on  $(Y_k, F_k)$  is  $\Theta_k = \{\theta_k \in P(\Theta_k): \theta_k \ll \nu_k\}$ , where  $\nu_k$  is a  $\sigma$ -finite measure on  $(Y_k, B(Y_k))$  with non-finite support. The reward function  $r_k: Y_k \rightarrow \mathbb{R}$  is a bounded, measurable function with  $\text{ess inf } r_k(y_k) = \sup\{a \in \mathbb{R}: \nu_k(\{y_k: r_k(y_k) < a\}) = 0\} = b_k$ , and  $\text{ess sup } r_k(y_k) = \inf\{a \in \mathbb{R}: \nu_k(\{y_k: r_k(y_k) > a\}) = 0\} = c_k$ . We define  $\|r_k\| = \max\{|b_k|, |c_k|\}$ .

With  $d_k$  the  $L^1$  metric on  $\Theta_k$ ,  $(\Theta_k, d_k)$  is a complete, separable metric space. For  $\theta_k \in \Theta_k$ , the product measure  $\theta_k \times \theta_k \times \dots$  on  $(\Omega_k, F_k)$  is denoted by  $P_{k, \theta_k}$ . The product space  $(\Theta, d_\Theta)$  is defined by  $\Theta = \Theta_1 \times \Theta_2$  where  $d_\Theta$  is a metric that metrizes the product topology. The Borel  $\sigma$ -field is  $B(\Theta) = B(\Theta_1) \times B(\Theta_2)$ . For  $\theta = (\theta_1, \theta_2) \in \Theta$ , the product probability measure  $P_\theta$  on  $(\Omega, F)$  is defined by  $P_\theta = P_{1, \theta_1} \times P_{2, \theta_2}$ .

In order to make use of Gittens Index machinery, we require that observing an outcome from arm  $k$  provides no information regarding the probability law governing arm  $k'$ , for  $k \neq k'$ . More formally the prior belief must be a product probability on  $(\Theta, B(\Theta))$ . Define  $\Pi(\Theta) = \{\mu \in P(\Theta): \mu = \mu_1 \times \mu_2, \text{ s.t. } \mu_1 \in P(\Theta_1), \mu_2 \in P(\Theta_2)\}$ . Identifying  $\mu = \mu_1 \times \mu_2 \in \Pi(\Theta)$  by the vector  $(\mu_1, \mu_2)$  of marginals, no confusion should arise from writing  $(\mu_1, \mu_2)$  instead of  $\mu_1 \times \mu_2$ . The projection functions  $\rho_k: \Pi(\Theta) \rightarrow P(\Theta_k)$  are defined by  $\rho_k((\mu_1, \mu_2)) = \mu_k$ .

Corresponding to a prior  $\mu$  are induced probability measures  $P_\mu$  and  $Q_\mu$  on respectively  $(\Omega, F)$  and  $(\Theta \times \Omega, B(\Theta) \times F)$ . For  $\mu = (\mu_1, \mu_2) \in \Pi(\Theta)$ ,  $Q_\mu$  is defined on measurable rectangles by  $Q_\mu(A \times B) =$

$\int_A P_{\theta(B)} \mu(d\theta)$  for  $A \in B(\Theta)$  and  $B \in F$ . By the Product Measure Theorem (see Ash (1972, Theorem 2.6.2)) there is

a unique extension onto  $B(\Theta) \times F$ .  $P_\mu$  is defined by  $P_\mu(B) = P_\mu(\Theta \times B)$  for every  $B \in F$ . From the perspective of the Bayesian decision-maker who has prior  $\mu$ , the relevant probability spaces are  $(\Omega, F, P_\mu)$  and  $(\Theta \times \Omega, B(\Theta) \times F, Q_\mu)$ . But from the perspective of a classical statistician, the probability space on which all random variables are defined is  $(\Omega, F, P_\theta)$  where  $\theta = (\theta_1, \theta_2) \in \Theta$  is the "true" parameter.

#### 4.3. Histories and Policies

We now provide a precise statement of the optimization problem from the perspective of the decision-maker. We start by defining an admissible plan. It will be convenient to include in our definition, "count functions"

$C_k(t): \Omega \rightarrow \{0, 1, 2, \dots\}$ , for  $k = 1, 2$  and  $t = 0, 1, 2, \dots$ . The realization  $C_k(t)(\omega)$  will be interpretable as the number of occasions that arm  $k$  has been chosen through time  $t$ .

The indicator function  $I_k: X \rightarrow \{0, 1\}$  is defined by  $I_k(x) = 1$  iff  $x = x_k$ . An *admissible plan* is a tuple  $[\{X(t)\}, \{Y(t)\}, \{C_1(t)\}, \{C_2(t)\}, \{H_t\}]$  where for  $t = 0, 1, 2, \dots$ ,  $X(t): \Omega \rightarrow X$ ,  $Y(t): \Omega \rightarrow Y$ ,  $C_k(t): \Omega \rightarrow \{0, 1, \dots\}$ , and  $\{H_t\}$  is a sequence of sub- $\sigma$ -fields of  $F$ , such that:

- i)  $H_0 = \{\emptyset, \Omega\}$
- ii)  $X(t): \Omega \rightarrow X$  is  $H_t$  measurable;
- iii) for  $t \geq 1$ ,  $H_t = H_{t-1} \vee \sigma(X(t-1), Y(t-1))$ ,
- iv)  $C_k(0) = I_k(X(0))$ ,
- v)  $C_k(t) = C_k(t-1) + I_k(X(t))$  for  $t \geq 1$ , and
- vi) if  $X(t)(\omega) = x_k$  and  $C_k(\omega) = n$ , then  $Y(t)(\omega_1, \omega_2) = \pi_k(n, \omega_k)$ .

For  $\alpha \in (0, 1)$ , an admissible plan  $[\{X(t)\}, \{Y(t)\}, \{C_1(t)\}, \{C_2(t)\}, \{H_t\}]$  is  $(\mu, \alpha)$ -*optimal* if for any other admissible plan  $[\{X(t)'\}, \{Y(t)'\}, \{C_1(t)'\}, \{C_2(t)'\}, \{H_t'\}]$ :

$$\int_{\Omega} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} \cdot r(X(t)(\omega), Y(t)(\omega)) \right] P_{\mu}(d\omega) \geq \int_{\Omega} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} \cdot r(X(t)'(\omega), Y(t)'(\omega)) \right] P_{\mu}(d\omega).^2$$

#### 4.4. Reformulation as a Bayesian Dynamic Programming Problem

In this section we recast the decision-maker's optimization problem as a Bayesian dynamic programming problem with the state at time  $t$  being the Bayesian's posterior probability  $\mu(t) = (\mu_1(t), \mu_2(t)) \in \Pi(\Theta)$ . As in Blackwell (1965), a *dynamic programming problem* is defined as a quintuple  $[X, u, \Pi(\Theta), \phi, \alpha]$ , where the *action space*  $X$ , the *discount factor*  $\alpha \in [0, 1)$  and the *state space*  $\Pi(\Theta)$  are as previously defined. The expected reward function for arm  $k$  is  $u_k: P(\Theta_k) \rightarrow \mathbb{R}$  given by  $u_k(\mu_k) = \int_{\Theta_k} \int_{Y_k} r_k(y_k) \theta_k(dy_k) \mu_k(d\theta_k)$ . The *expected reward function*  $u$ :

$\Pi(\Theta) \times X \rightarrow \mathbb{R}$ , is defined by  $u((\mu_1, \mu_2), x_k) = u_k(\mu_k)$ .

To define the *transition probability*  $\phi: \Pi(\Theta) \times X \rightarrow P(\Pi(\Theta))$ , we need to develop some notation for Bayesian updating. Analogous to the definition of  $\Gamma$  in Section III, the arm  $k$  updating maps  $\Gamma_k: P(\Theta_k) \times Y_k \rightarrow P(\Theta_k)$ ,  $k = 1, 2$ , are chosen so that: (i)  $\Gamma_k$  is jointly measurable, and (ii) for each  $\mu_k \in P(\Theta_k)$ ,  $\Gamma_k(\mu_k, \cdot)$  is a regular version of conditional probability. The updating maps  $\Gamma_{kn}: P(\Theta_k) \times \Omega_k \rightarrow P(\Theta_k)$  are recursively defined by: (i)  $\Gamma_{k1}(\mu_k, \omega_k) = \Gamma_k(\mu_k, \pi_k(1, \omega_k))$  and, (ii) for  $n > 1$ ,  $\Gamma_{kn}(\mu_k, \omega_k) = \Gamma_k(\Gamma_{k,n-1}(\mu_k, \omega_k), \pi_k(n, \omega_k))$ .

Given that action  $x_k$  is chosen with prior marginal belief  $\mu_k$ , the Bayesian's probability distribution over next period marginal posterior beliefs is  $\Psi_k(\mu_k) \in P^2(\Theta)$ , where the map  $\Psi_k: P(\Theta_k) \rightarrow P^2(\Theta_k)$  is constructed as follows. For  $\mu_k \in P(\Theta_k)$  and  $B \in B(P(\Theta_k))$ , define  $v(\mu_k, B) = \Gamma_k^{-1}(\mu_k, \cdot)(B)$ . Since  $v(\mu_k, B) \in B(Y_k)$ , the map

$\theta_k \rightarrow \theta_k(v(\mu_k, B))$  is measurable and  $\Psi_k(\mu_k)(B) = \int_{\Theta_k} \theta_k(v(\mu_k, B)) \mu_k(d\theta_k)$ . We can now define the transition

probability  $\varphi: \Pi(\Theta) \times X \rightarrow P(\Pi(\Theta))$  by:

$$\varphi((\mu_1, \mu_2), x_1)(A_1 \times A_2) = \Psi_1(\mu_1)(A_1) \cdot I_{A_2}(\mu_2),$$

and

$$\varphi((\mu_1, \mu_2), x_2)(A_1 \times A_2) = \Psi_2(\mu_2)(A_2) \cdot I_{A_1}(\mu_1),$$

for  $A_1 \in B(P(\Theta_1))$ ,  $A_2 \in B(P(\Theta_2))$  and  $I_{A_k}(\cdot)$  the indicator function.

A *policy* is a sequence  $\{X(t)\}$  such that there exists a (unique) admissible plan  $[\{X(t)\}, \{Y(t)\}, \{C_1(t)\}, \{C_2(t)\}, \{H_t\}]$ . The set of policies is denoted by  $\Xi$ . Given  $\{X(t)\} \in \Xi$  and prior belief  $\mu(0) = (\mu_1(0), \mu_2(0)) \in \Pi(\Theta)$ , there is an induced sequence  $\{\mu(t)\} = \{\mu_1(t), \mu_2(t)\}$  of *state variables* (posterior beliefs), where  $\mu_k(t): \Omega \rightarrow P(\Theta_k)$ . For  $t \geq 1$ ,  $\mu_k(t)$  is defined by: (i) if  $X(t-1)(\omega) = x_k$ , then  $\mu_k(t)(\omega) = \Gamma_k(\mu_k(t-1), \pi_k(C_k(t), \omega))$ , and (ii) if  $X(t-1)(\omega) \neq x_k$ , then  $\mu_k(t)(\omega) = \mu_k(t-1)(\omega)$ .

For  $\alpha \in (0, 1)$  and  $\mu(0) \in \Pi(\Theta)$ , a policy  $\{X(t)\}$  with corresponding posterior sequence  $\{\mu(t)\}$  is  $(\mu(0), \alpha)$  *DP-optimal* if for any policy  $\{X'(t)\}$  with corresponding posterior sequence  $\{\mu'(t)\}$ :

$$\int_{\Omega} \sum_{t=0}^{\infty} \alpha^t \cdot u(\mu(t)(\omega), X(t)(\omega)) P_{\mu(0)}(d\omega) \geq \int_{\Omega} \sum_{t=0}^{\infty} \alpha^t \cdot u(\mu'(t)(\omega), X'(t)(\omega)) P_{\mu(0)}(d\omega).$$

A seemingly obvious, but non-trivial fact is that for  $\alpha \in (0, 1)$  an admissible plan  $[\{X(t)\}, \{Y(t)\}, \{C_1(t)\}, \{C_2(t)\}, \{H_t\}]$  is  $(\mu, \alpha)$  optimal (as defined in Section 4.3) iff  $\{X(t)\}$  is  $(\mu, \alpha)$  DP-optimal. Formally, this result follows from Theorem 7.3 of Reider (1975).

For  $\alpha = 0$ , the above definition of optimality would not be useful since it would impose no restrictions on behavior after period 0. Associating  $\alpha = 0$  with repeatedly myopic behavior, we define a policy  $\{X(t)\}$  with corresponding posterior sequence  $\{\mu(t)\}$  to be  $(\mu(0), 0)$  DP-optimal if for all  $t$ ,  $u(\mu(t), X(t)) \geq u(\mu(t), x_k)$ ,  $P_{\mu(0)}$  a.s..

A policy  $\{X(t)\}$  with associated posterior maps  $\{\mu_1(t)\}$ ,  $\{\mu_2(t)\}$  is *stationary* if there exists a *policy function*  $\xi: \Pi(\Theta) \rightarrow X$  such that  $\xi$  is measurable and for all  $t$ ,  $X(t) = \xi(\mu_1(t), \mu_2(t))$ . If  $\{X(t)\}$  is stationary and DP-optimal, then the policy function  $\xi$  is an *optimal policy function*. Applying standard results in dynamic programming (see e.g., Blackwell (1965) or Maitra (1968)) it is routine to verify that an optimal, stationary policy exists.

We now restate some standard dynamic programming results in the context of our model. For discount factor  $\alpha$ , the *value function*  $V: P(\Theta) \rightarrow \mathbb{R}$  is defined by

$$V(\mu) = \sup_{\{X(t)\} \in \Xi} \left\{ \int_{\Omega} \sum_{t=0}^{\infty} \alpha^t \cdot u(\mu(t)(\omega), X(t)(\omega)) P_{\mu}(d\omega) \right\}$$

where  $\{\mu(t)\}$  denotes the posterior sequence corresponding to  $\{X(t)\}$ .  $V$  satisfies the Bellman or optimality equation (see e.g. Blackwell (1965, Theorem 6 (e))):

$$V(\mu) = \text{Max}_{x \in X} \left\{ u(\mu, x) + \alpha \cdot \int_{\Pi(\Theta)} V(\mu') \varphi(\mu, x)(d\mu') \right\}.$$

Additionally,  $\xi$  is an optimal policy function iff it solves the optimality equation (Blackwell [1965, Theorem 6 (f)]). That is for all  $\mu \in \Pi(\Theta)$ :

$$V(\mu) = u(\mu, \xi(\mu)) + \alpha \cdot \int_{\Pi(\Theta)} V(\mu') \phi(\mu, \xi(\mu))(d\mu').$$

#### 4.5. Gittens Index Characterization of Optimal Policy

There is a simple characterization of optimal policies, based upon initial results of Gittens and Jones (1974), and subsequent refinements by Berry and Fristedt (1985), Ross (1983) and Whittle (1982). Gittens and Jones proved the existence of functions  $M_1: P(\Theta_1) \rightarrow \mathbb{R}$  and  $M_2: P(\Theta_2) \rightarrow \mathbb{R}$  with the property that it is optimal to choose arm 1 in period  $t$  iff  $M_1(\mu_1(t)) \geq M_2(\mu_2(t))$ . The function  $M_k$  is commonly referred to as the *Gittens Index* for arm  $k$ .

To motivate the definition of  $M_k$  consider a one-armed bandit problem where in the initial stage the decision-maker has the option of playing arm 1 or stopping and collecting a terminal reward of  $m$ . In subsequent stages, assuming arm 1 has been played in all previous stages, the options remain selecting arm 1 or stopping and receiving a final payment  $m$ . The value of the Gittens Index for belief  $\mu_k$  is the terminal reward  $m$  such that the decision-maker is indifferent between continuing and stopping.

More precisely, denote the set of stopping times on  $(\Omega_k, F_k)$  as  $\mathcal{T}_k = \{\tau: \Omega_k \rightarrow \mathbb{N}_0, \tau^{-1}(\{n\}) \in F_{kn}\}$ . The expected total reward of the stopping time policy  $\tau$  with terminal payoff  $m$  and belief  $\mu_k \in P(\Theta_k)$  is given by the function  $T_k: \mathcal{T}_k \times P(\Theta_k) \times \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$T_k(\tau, \mu_k, m) = \int_{\Omega_k} \left[ \sum_{t=0}^{\tau(\omega_k)-1} \alpha^t \cdot r_k(\omega_{k,t+1}) \right] + \alpha^{\tau(\omega_k)} \cdot m \cdot P_{k, \mu_k}(d\omega_k).$$

The value of the optimal policy is  $V_k(\mu_k, m) = \sup_{\tau \in \mathcal{T}_k} \{T_k(\tau, \mu_k, m)\}$ . Recalling that  $c_k = \text{ess sup}\{r_k(y_k)\}$ , it is routine to verify that for all  $\mu_k \in P(\Theta_k)$ ,  $V_k(\mu_k, m) = m$  for all  $m \geq c_k \cdot (1 - \alpha)^{-1}$ . The Gittens Index is defined by  $M_k(\mu_k) = \inf\{m: V_k(\mu_k, m) = m\}$ . A characterization of the optimal policy in terms of the Gittens Index is given by the following proposition.

**PROPOSITION 4.1.** *A policy  $\{X(t)\}$  with posterior maps  $\{\mu_1(t)\}$ ,  $\{\mu_2(t)\}$  is  $(\mu_1(0), \mu_2(0))$ -optimal iff  $X(t)(\omega) = x_k$  whenever  $M_k(\mu_k(t)(\omega)) > M_j(\mu_j(t)(\omega))$ .*

*Proof.* Whittle (1982, Theorem 14.4.1). ■

Throughout the remainder of the paper we assume that the Bayesian controller follows an optimal stationary policy  $\{X(t)\}$  with posterior maps  $\{\mu_1(t)\}$ ,  $\{\mu_2(t)\}$ , and policy function  $\xi: \Pi(\Theta) \rightarrow X$  defined by  $\xi((\mu_1(t), \mu_2(t))) = x_1$  iff  $M_1(\mu_1(t)) \geq M_2(\mu_2(t))$ . For a more detailed development of the Gittens Index and the optimality of the Index policy, the texts of Ross (1983), Whittle (1982), and Berry and Fristedt (1985) are recommended.

## 5. GENERIC LIMIT THEOREMS

### 5.1. Some Continuity Results

Preparatory to proving the genericity theorems, some preliminary technical results are developed in this subsection. The principal result is the establishment of the continuity of the functions  $M_1$  and  $M_2$ . The first step is to develop characterizations of  $V_1$  and  $V_2$  by making use of the fact that these value functions are solutions to the optimality equation.

LEMMA 5.1. *The function  $V_k: P(\Theta_k) \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous, for  $k = 1, 2$ .*

*Proof.* Without loss of generality we consider the case of  $k = 1$ . We first note that an easy consequence of Lemma A.5. is that  $\Psi_1$  is continuous. Let  $C(P(\Theta_1))$  denote the space of bounded, continuous real-valued functions on  $P(\Theta_1)$  endowed with the sup norm. Since  $u(\mu_1, \mu_2, x_1)$  is independent of  $\mu_2$ , we may define  $u_1(\mu_1) = u(\mu_1, \mu_2, x_1)$ . For each  $m \in \mathbb{R}$ , define the mapping  $\vartheta_m: C(P(\Theta_1)) \rightarrow C(P(\Theta_1))$  by  $\vartheta_m(\zeta)(\mu_1) = \max\{m, u_1(\mu_1) + \alpha \cdot \int \zeta(\gamma_1) \Psi_1(\mu_1)(d\gamma_1)\}$ . (The continuity of  $\vartheta_m(\zeta(\cdot))$  follows from the continuity of  $u_1$  and  $\Psi_1$ .) By standard arguments one can verify that  $\vartheta_m$  is a contraction mapping with modulus  $\alpha$ . Therefore the fixed-point of  $\vartheta_m$  is a continuous function of  $m$ . Since  $V_1(\cdot, m)$  is the fixed-point of  $\Xi_m$ , the map  $(\mu_1, m) \rightarrow V_k(\mu_1, m)$  is continuous. ■

LEMMA 5.2. *For  $k = 1, 2$ , and  $\mu_k \in P(\Theta_k)$ , the map  $m \rightarrow [V_k(\mu_k, m) - m]$  is decreasing.*

*Proof.* Replacing sums with integrals, the proof of Ross (1983, Lemma VII.2.1) remains valid. ■

Recall that  $\Psi_k(\mu_k)$  is the Bayesian's probability distribution over next period beliefs on  $\Theta_k$ , given that arm  $k$  is chosen and  $\mu_k$  is the current belief. Define  $W_k: P(\Theta_k) \times \mathbb{R} \rightarrow \mathbb{R}$  by  $W_k(\mu_k, m) = u_k(\mu_k) + \alpha \cdot \int_{P(\Theta_k)} V_k(\mu_k', m) \Psi_k(\mu_k)(d\mu_k')$ .  $W_k(\mu_k, m)$  is the expected reward for the one-armed bandit problem with the option of stopping and receiving a payoff of  $m$  after the initial period.

LEMMA 5.3. (i)  $W_1$  and  $W_2$  are continuous. (ii)  $M_k(\mu_k) < m$  iff  $W_k(\mu_k, m) < m$ .

(i) This follows from the continuity of  $\Psi_k$  and Lemma 5.1.

(ii) Suppose  $W_k(\mu, \bar{m}) < \bar{m}$ . By definition of  $W_k$ ,  $\bar{m} > u_k(\mu) + \alpha \cdot \int V_k(\mu', \bar{m}) \Psi_k(\mu)(d\mu')$ . Since  $V_k$  is continuous (Lemma 5.1) and increasing in  $m$ ,  $(\bar{m} - \varepsilon) > u_k(\mu) + \alpha \cdot \int V_k(\mu', \bar{m} - \varepsilon) \Psi_k(\mu)(d\mu')$  for  $\varepsilon$  sufficiently small. But this last inequality is, by definition, equivalent to  $M_k(\mu) < \bar{m} - \varepsilon < \bar{m}$ .

For the converse, suppose  $M_k(\bar{\mu}) = \bar{m} < m$ . Since  $\bar{m}$  is a terminal payoff for which the decision-maker is indifferent between continuing play and quitting,  $W_k(\bar{\mu}, \bar{m}) = V_k(\bar{\mu}, \bar{m}) = \bar{m}$ . From Lemma 5.2, for all  $\mu \in P(\Theta_k)$ ,



$V_k(\mu, m) - V_k(\mu, \bar{m}) \leq (m - \bar{m})$ , implying that  $\int [V_k(\mu', m) - V_k(\mu', \bar{m})] \phi_k(\bar{\mu})(d\mu') \leq (m - \bar{m})$ . Finally, we have that:

$$\begin{aligned}
 W_k(\bar{\mu}, m) &= u_k(\bar{\mu}) + \alpha \cdot \int V_k(\mu', m) \Psi_k(\bar{\mu})(d\mu') \\
 &= u_k(\bar{\mu}) + \alpha \cdot \left\{ \int V_k(\mu', m) \Psi_k(\bar{\mu})(d\mu') + \int V_k(\mu', \bar{m}) \Psi_k(\bar{\mu})(d\mu') - \int V_k(\mu', \bar{m}) \Psi_k(\bar{\mu})(d\mu') \right\} \\
 &= \{u_k(\bar{\mu}) + \alpha \cdot \int V_k(\mu', \bar{m}) \Psi_k(\bar{\mu})(d\mu')\} + \alpha \cdot \left\{ \int V_k(\mu', m) \Psi_k(\bar{\mu})(d\mu') - \int V_k(\mu', \bar{m}) \Psi_k(\bar{\mu})(d\mu') \right\} \\
 &= \bar{m} + \alpha \cdot \left\{ \int V_k(\mu', m) \Psi_k(\bar{\mu})(d\mu') - \int V_k(\mu', \bar{m}) \Psi_k(\bar{\mu})(d\mu') \right\} \\
 &\leq \bar{m} + \alpha \cdot (m - \bar{m}) \\
 &< m. \blacksquare
 \end{aligned}$$

PROPOSITION 5.4.  $M_1$  and  $M_2$  are continuous.

*Proof.* First we establish that  $M_k$  is lower-semicontinuous by verifying that for all  $c \in \mathbb{R}$  the set  $\{\mu \in P(\Theta_k): M_k(\mu) > c\}$  is open. Suppose that  $M_k(\bar{\mu}) > c$ . By definition of  $M_k$ ,  $V_k(\bar{\mu}, c) - c > 0$ . From the continuity of  $V_k$  (Lemma 5.1),  $\exists$  an open neighborhood  $N$  of  $\bar{\mu}$  such that for all  $\mu \in N$ ,  $V_k(\bar{\mu}, c) - c > 0$ . Therefore, for all  $\mu \in N$ ,  $V_k(\mu, c) - c > 0$  and so  $M_k(\mu) > c$ .

Upper-semicontinuity is confirmed by demonstrating that the set  $\{\mu \in P(\Theta_k): M_k(\mu) < c\}$  is open for all  $c \in \mathbb{R}$ . Suppose that  $M_k(\bar{\mu}) = \bar{m} < c$ . By Lemma 5.3,  $W_k(\bar{\mu}, c) < c$  and  $\exists$  an open neighborhood  $J$  of  $\bar{\mu}$  such that for  $\mu \in J$ ,  $W_k(\mu, c) < c$ . But this implies that  $M_k(\mu) < c$ .  $\blacksquare$

## 5.2. Generic Outcomes when $b_1 \neq b_2$

In this subsection we analyze the limit behavior of the decision-maker for the case where  $b_1 \neq b_2$  (recall that  $b_k = \text{ess inf}\{\pi_k(y_k): y_k \in Y_k\}$ ). Without loss of generality we assume that  $b_1 < b_2$ . To motivate the next result, choose a set  $Y_0 \subset Y_1$  with  $v_1(Y_0) > 0$  and  $\sup\{r_1(y_1): y_1 \in Y_0\} < b_2$ . So if at time  $t$ , the decision-maker's conditional probability of an outcome in  $Y_0$  occurring if arm 1 is selected is sufficiently large, arm 2 will be selected regardless of  $\mu_2(t)$ ; and so arm 2 will be selected at all times  $t' \geq t$ . From Theorem 3.10 we can conclude that there is a residual set  $\mathcal{R}_1 \subset \Theta_1 \times P(\Theta_1)$  with the following property. If  $(\theta_1, \mu_1) \in \mathcal{R}_1$ ,  $\mu_1$  is the prior belief on  $(\Theta_1, B(\Theta_1))$ , and arm 1 is played sufficiently often, the decision-maker will eventually,  $P_{\theta_1}$  a.s., become sufficiently pessimistic that arm 1 will never be tested again. Consequently, arm 1 will be played only finitely often.

To formalize the above remarks, we begin by defining  $Y_0$  as above. Choose  $\mu_{10} \in P(\Theta_1)$  such that  $\mu_{10}(Y_0) = 1$ . For  $\varepsilon > 0$  define an open neighborhood  $G_\varepsilon \subset P(\Theta_1)$  of  $\mu_{10}$ , by  $G_\varepsilon = \{\mu_1 \in P(\Theta_1): \beta(\mu_1, \mu_{10}) < \varepsilon\}$ .

LEMMA 5.5. *There exists  $c > 0$ , such that for all  $\varepsilon < c$ ,  $\sup_{\mu_1 \in G_\varepsilon} \{M_1(\mu_1)\} < \inf_{\mu_2 \in P(\Theta_2)} \{M_2(\mu_2)\}$ .*

*Proof.* First observe that: (i)  $\inf\{M_2(\mu_2): \mu_2 \in P(\Theta_2)\} = \frac{b_2}{1-\alpha}$ , and (ii)  $M_1(\mu_{10}) < \frac{b_2}{1-\alpha}$ . Since  $M_1$  is continuous,  $J = \{\mu_1 \in P(\Theta_1): M_1(\mu_1) < \frac{b_2}{1-\alpha}\}$  is open. So  $\exists c$  such that for  $\varepsilon < c$ ,  $G_\varepsilon \subset J$ . And so for  $\mu_1 \in G_\varepsilon \subset J$ , and any  $\mu_2 \in P(\Theta_2)$ ,  $M_1(\mu_1) < \frac{b_2}{1-\alpha} \leq M_2(\mu_2)$ . ■

**PROPOSITION 5.6.** *Suppose  $b_1 < b_2$ . Then there exists a residual subset  $\mathcal{R}_1 \subset \Theta_1 \times P(\Theta_1)$  such that for  $(\theta_1, \mu_1(0)) \in \mathcal{R}_1$  and all  $(\theta_2, \mu_2(0)) \in \Theta_2 \times P(\Theta_2)$ ,  $P_\theta$  a.s.  $X(t)(\omega) = x_1$  only finitely often.*

*Proof.* Invoking Lemma 5.5, choose an open set  $G_\varepsilon$  such that  $M_1(\mu_1) < M_2(\mu_2)$  for all  $(\mu_1, \mu_2) \in G_\varepsilon \times P(\Theta_2)$ . Define the set  $\Omega_{1\varepsilon} \subset \Omega_1$  by  $\Omega_{1\varepsilon} = \{\omega_1 \in \Omega_1: \Gamma_{1n}(\mu_1(0), \omega_1) \notin G_\varepsilon, \text{ for all } n = 1, 2, \dots\}$ . Define  $\mathcal{R}_1 = \{(\theta_1, \mu_1(0)) \in \Theta_1 \times P(\Theta_1): P_{\theta_1}(\Omega_{1\varepsilon}) = 0\}$ . By Theorem 3.10,  $\mathcal{R}_1$  is residual. ■

### 5.3. Generic Outcomes when $b_1 = b_2$ .

When  $b_1 = b_2$ , the “typical” outcome will be that the decision-maker will start out playing one arm, but will eventually become sufficiently pessimistic regarding the first choice and switch to the other arm. Eventually, though she will become sufficiently regarding the non-initial arm and switch back to the original arm. This switching back and forth will continue forever.

**PROPOSITION 5.7.** *Suppose  $b_1 = b_2 = b$  and  $v_k(\{y_k: r_k(y_k) \neq b\}) > 0$ . Then there is a residual set  $R \subset \Theta \times \Pi(\Theta)$  such that  $P_\theta$  a.s.,  $X(t) = x_1$  infinitely often and  $X(t) = x_2$  infinitely often.*

*Proof.* Let  $R_k = \{(\theta_k, \mu_k(0)) \in \Theta_k \times P_k(\Theta): P_{\theta_k} \text{ a.s. } \limsup_n \Gamma_{kn}(\mu_k(0), \omega_k)(G) = 1, \text{ for all open } G \subset \Theta_k\}$ . By Theorem 3.10,  $R_k$  is residual, and so  $R = R_1 \times R_2$  is residual. Now choose  $(\theta_1, \mu_1(0), \theta_2, \mu_2(0)) \in R$ , and define  $E_1 = \{\omega: \sup_t C_1(t)(\omega) \text{ is finite}\}$ .  $E_1$  is the  $\omega$ -set for which arm 1 is played only finitely, given the optimal policy starting from beliefs  $(\mu_1(0), \mu_2(0))$ . Because of the symmetry of the specification, it suffices to prove that  $P_\theta(E_1) = 0$ . For  $\omega \in E_1$ , define the terminal value of the Gittens index for arm 1 as  $m_{1\infty}(\omega) = M_1(\Gamma_{1, C_1(t)(\omega)}(\mu_1(0), \omega_1))$ . Since  $(\theta_1, \mu_1(0)) \in R_1$ ,  $P_\theta(E_1 \cap \{\omega: m_{1\infty}(\omega) = \frac{b}{1-\alpha}\}) = 0$ . Now let  $E_2 = \{(\omega_1, \omega_2): \limsup_n \Gamma_{2n}(\mu_2(0), \omega_2)(G) = 1, \text{ for all open } G \subset \Theta_2\}$ , and by Theorem 3.10,  $P_\theta(E_2) = 1$ . By the continuity of  $M_2$ ,  $E_1 \subset \{E_1 \cap \{\omega: m_{1\infty}(\omega) = \frac{b}{1-\alpha}\}\} \cup \sim E_2$ , and so  $P_\theta(E_1) = 0$ . ■

## 6. CONCLUDING REMARKS

In Section 3, we demonstrated the asymptotic sensitivity of posterior beliefs with respect to a prior probability on a parameter space which is not finite dimensional. These results have significance for Bayesian statistical decision theory. Since posterior beliefs are not robust to small perturbations of the prior, the optimal action correspondence is similarly non-robust. If two Bayesians have identical objective or loss functions, nearby prior beliefs *and observe the same sequence of outcomes*, without additional restrictions, it would not be pathological for them to each choose

actions that the other evaluated as markedly inferior to their own choice. The bandit example, developed in Sections 4 and 5 of this paper, illustrates this phenomena in a context where beliefs do, in fact, affect outcomes.

For stochastic processes that might be generated by more complex feedback between beliefs and actions, such as rational expectations models with learning, no formal demonstration is provided of generic nonconvergence of beliefs and outcomes. Nevertheless, it would be surprising (at least to this author) if the additional complexity of such models, somehow restored the asymptotic regularity attained in analyses conducted from the perspective of the Bayesian learner.

Economic theorists who wish to model learning by Bayesian economic agents are confronted with three options. One alternative is to impose no restrictions upon agent beliefs and analyze the resulting stochastic process from the perspective of a classical statistician who knows the true parameter. With sufficiently specific assumptions on other parameters of the model (such as preferences and technology) it may be possible show that agents with consistent priors will eventually be financially dominant<sup>3</sup>, and presumably then the limiting price process would be indistinguishable from the price process emanating from a model in which all agents had consistent priors. Even so, there would remain the question as to whether any such asymptotic properties would be robust under arbitrarily small perturbations of parameter values and initial beliefs.

A second alternative would be to follow the currently predominant practice of adopting the probabilistic vantage point of the Bayesian agent(s), imposing no restrictions on agent beliefs. A drawback with this approach is that it provides no basis for drawing distributional inferences for any particular parameter value of agent prior probability zero. The resulting limit theorems can be interpreted as predictions by the economic theorist only if the theorist's beliefs are absolutely continuous with respect to the agent's beliefs. So unless the reader also has beliefs absolutely continuous with respect to the agent's, there are no grounds for accepting the agent's asymptotic predictions.<sup>4</sup>

The final option, one that I endorse, is to narrow the set of candidate prior beliefs, a strategy that is adopted by Bikhchandani and Sharma (1990). To motivate this strategy, consider the bandit problem studied in Sections 4 and 5, and for simplicity suppose that  $Y_1$  and  $Y_2$  are each countable. If I was the decision-maker, I might find it difficult to exactly specify my prior, but I would reject any prior belief for which there was a residual  $\theta$ -set  $A$ , such that for all  $\theta \in A$ , with  $P_\theta$  probability one my beliefs would not converge. In particular, I would require that for any arm played infinitely often, the Prohorov (or bounded Lipschitz) distance between my posterior beliefs and the sample distribution converge to zero. And while acknowledging that introspective reasoning has its limitations, I believe that few individuals would behave as predicted by Propositions 5.6 and 5.7. More generally, in environments where consistent estimators are available, the modeller should assume that priors are chosen from the family of probability measures that are consistent for all  $\theta \in \Theta$ .

This is philosophically similar to the "what if" method advocated by Diaconis and Freedman (1986) for Bayesian statisticians. Diaconis and Freedman suggest that "... after specifying a prior distribution, generate imaginary data sequences, compute the posterior, and consider whether the posterior would be an adequate representation of the updated prior." Adapting this recommendation to the context of economic modelling, it would be natural to require that agents have prior beliefs with full support and that the sequence of agent posterior beliefs

converges almost surely with respect to the measure  $P_\theta$  for all  $\theta \in \Theta$ . The feasibility of such a strategy requires the existence of consistent priors. Unfortunately, endogenous learning will typically imply a non-stationary stochastic process. And there are few results currently available on the  $P_\theta$  consistency (or convergence properties) of Bayes estimates for such processes.<sup>5</sup> Further research on sufficient conditions on priors for  $P_\theta$  consistency is needed.

## Appendix

For simplicity we explicitly prove Proposition 3.5 for  $n = 1$ . The argument for the  $n$ -fold product space is essentially identical and is briefly sketched.

We start by defining for  $A \in B(Z)$ , the function  $f_A: \Lambda \rightarrow \mathbb{R}$  by  $f_A(\lambda) = \int_A h(\lambda, z) \nu(dz)$ .

LEMMA A.1.  $f_A$  is a bounded Lipschitz function, and  $\|f_A\|_{BL} \leq 3$ .

*Proof.* The Lipschitz norm is defined by  $\|f_A\|_L = \sup_{\lambda \neq \lambda'} \{|f_A(\lambda) - f_A(\lambda')| / d_1(\lambda, \lambda')\}$ . Since

$$|f_A(\lambda) - f_A(\lambda')| \leq \int_A |h(\lambda, z) - h(\lambda', z)| \nu(dz) \leq 2 \cdot d_1(\lambda, \lambda') \leq 2,$$

the Lipschitz norm exists and  $\|f_A\|_L \leq 2$ . Since  $\|f_A\|_\infty = \sup_\lambda |f_A(\lambda)| = 1$ , the bounded Lipschitz norm  $\|f_A\|_{BL} = \|f_A\|_L + \|f_A\|_\infty$  exists and  $\|f_A\|_{BL} \leq 3$ . ■

LEMMA A.2. For  $\mu_i, \mu \in P(\Lambda)$  if  $\mu_i \Rightarrow \mu$  then  $\{\sup_A |\int_A f_A d(\mu_i - \mu)|: A \in B(Z)\} \rightarrow 0$ .

*Proof.* Recall that the Dudley metric  $\beta$  is defined by  $\beta(\mu, \gamma) = \sup\{|\int f d(\mu - \gamma)|: \|f\|_{BL} \leq 1\}$  and that  $\mu_i \Rightarrow \mu$  iff  $\beta(\mu_i, \mu) \rightarrow 0$ . So by Lemma A.1,  $\mu_n \Rightarrow \mu$  implies  $\{\sup_A |\int_A f_A d(\mu_n - \mu)|: A \in B(Z)\} \rightarrow 0$ . ■

For  $\mu \in P(\Lambda)$  define the probability  $P_\mu$  on  $(Z \times \Lambda, B(Z) \times P(\Lambda))$  by  $P_\mu(A \times B) = \int_B f_A(\lambda) \mu(d\lambda)$ . Also define the marginal probability  $Q_\mu$  on  $(Z, B(Z))$  by  $Q_\mu(A) = P_\mu(A \times \Lambda)$ . Now fix a sequence  $\mu_i \Rightarrow \mu \in P(\Lambda)$  and define  $L_i(z) = \int_\Lambda h(\lambda, z) \mu_i(d\lambda)$  and  $L(z) = \int_\Lambda h(\lambda, z) \mu(d\lambda)$ .  $L_i(z)$ ,  $L(z)$  are the Radon-Nikodym derivatives of  $Q_{\mu_i}$  and  $Q_\mu$  with respect to  $\nu$ .

LEMMA A.3.  $L_i(\cdot) \rightarrow L(\cdot)$  in  $\nu$  measure.

$$\begin{aligned} \text{Proof. } \int_A L_i(z) \nu(dz) &= \int_A \int_\Lambda h(\lambda, z) \mu_i(d\lambda) \nu(dz) = \int_\Lambda \int_A h(\lambda, z) \nu(dz) \mu_i(d\lambda) \\ &= \int_\Lambda f_A(\lambda) \mu_i(d\lambda) \rightarrow \int_\Lambda f_A(\lambda) \mu(d\lambda) = \int_A L(z) \nu(dz). \end{aligned}$$

Furthermore, by Lemma A.2, the convergence is uniform in  $A$ . The conclusion follows from Remark 3.1 of [Strasser, 1985 #57]. ■

For  $B \in B(\Lambda)$ , define  $L_i(B, x) = \int_B h(\lambda, z) \mu_i(d\lambda)$  and  $L(B, z) = \int_B h(\lambda, z) \mu(d\lambda)$

LEMMA A.4. If  $B \in B(\Lambda)$  and  $\mu(\partial B) = 0$ , then  $L_i(B, z) \rightarrow L(B, z)$  in  $\nu$  measure.

*Proof.* First consider the case where  $\mu(B) > 0$ . It suffices to verify that  $|\int_A L_i(B, z) \nu(dz) - \int_A L(B, z) \nu(dz)| \rightarrow 0$  uniformly in  $A$ . Let  $B(B)$  denote the Borel sets restricted to  $B$ . Define  $\mu_i(B) = a_i$ ; and define  $\gamma_i$  on  $(B, B(B))$  by  $\gamma_i(C)$



$= a_i^{-1} \cdot \mu_i(C)$ , for  $C \in \mathcal{B}(B)$ . Similarly, define  $a = \mu(B)$  and define  $\gamma$  on  $(B, \mathcal{B}(B))$  by  $\gamma(C) = a^{-1} \cdot \mu(C)$ . Observe that  $\int_A L_i(B, z) v(dz) = \int_B f_A(\lambda) \mu_i(d\lambda)$  and that  $\int_A L(B, z) v(dz) = \int_B f_A(\lambda) \mu(d\lambda)$ . So

$$\begin{aligned} & \left| \int_A L_i(B, z) v(dz) - \int_A L(B, z) v(dz) \right| \\ &= \left| \int_B f_A(\lambda) \cdot a_i \gamma_i(d\lambda) - \int_B f_A(\lambda) \gamma(d\lambda) \right| \\ &\leq \left| \int_B f_A(\lambda) \cdot \alpha_i \gamma_i(d\lambda) - \int_B f_A(\lambda) \gamma_i(d\lambda) \right| + \left| \int_B f_A(\lambda) \gamma_i(d\lambda) - \int_B f_A(\lambda) \gamma(d\lambda) \right|. \end{aligned}$$

Since  $|f_A|$  is bounded,  $\left| \int_B f_A(\lambda) \cdot \alpha_i \gamma_i(d\lambda) - \int_B f_A(\lambda) \gamma_i(d\lambda) \right| \rightarrow 0$  independently of  $A$ . Substituting  $\gamma_i$  and  $\gamma$  for  $\mu_i$  and  $\mu$  in Lemma A.2.,  $\sup_{A \in \mathcal{B}(\Lambda)} \left| \int_B f_A(\lambda) \cdot a_i \gamma_i(d\lambda) - \int_B f_A(\lambda) \gamma(d\lambda) \right| \rightarrow 0$ .

If  $\mu(B) = 0$  then observe that  $L_i(z) = L_i(B, z) + L_i(\sim B, z)$  and  $L(z) = L_i(B, z) + L_i(\sim B, z)$ . Since  $B$  is a  $\mu$ -continuity set,  $\sim B$  is also a  $\mu$ -continuity set. So by application of the above and Lemma A.3,  $L_i(B, \cdot) \rightarrow L(B, \cdot)$  in  $v$  measure. ■

LEMMA A.5. Suppose  $\mu \in P(\Lambda)$  and pick  $\varepsilon > 0$ . Let  $\mathcal{A} = \{A_1, A_2, \dots\}$  be a disjoint cover of  $\Lambda$ , with diameter of  $A_t < \delta = \frac{\varepsilon}{5}$ . Define  $B_T = \bigcup_{t=1}^T A_t$ , and choose  $T$  so that  $\mu(B_T) > 1 - \delta$ . If  $\gamma \in P(\Lambda)$  and

$$\sup_{t \leq T} |\mu(A_t) - \gamma(A_t)| < \delta \cdot T^{-1}, \text{ then } \beta(\mu, \gamma) < \varepsilon.$$

*Proof.* Let  $g \in BL(\Lambda, d_1)$  with  $\|g\|_{BL} \leq 1$ . It suffices to prove that  $\left| \int_{\Lambda} g d\mu - \int_{\Lambda} g d\gamma \right| < \varepsilon$ . Define  $a_t = \inf\{g(\lambda) : \lambda \in A_t\}$ , define  $b_t = \sup\{g(\lambda) : \lambda \in A_t\}$ , and note that  $b_t - a_t < \delta$ . Since  $\int_{A_t} g d\gamma \geq [\mu(A_t) - \frac{\delta}{T}] \cdot a_t$ , and  $\int_{A_t} g d\mu \leq$

$$[\mu(A_t)] \cdot b_t, \text{ it follows that } \int_{A_t} g d\mu - \int_{A_t} g d\gamma \leq \delta \cdot \mu(A_t) + \frac{\delta}{T}.$$

$$\text{Similarly, } \int_{A_t} g d\gamma - \int_{A_t} g d\mu \leq \delta \cdot \mu(A_t) + \frac{\delta}{T}. \text{ So } \left| \int_{A_t} g d\gamma - \int_{A_t} g d\mu \right| \leq \delta \cdot \mu(A_t) + \frac{\delta}{T}.$$

$$\text{and } \left| \int_{B_T} g d\gamma - \int_{B_T} g d\mu \right| \leq \delta \cdot \mu(B_T) + \delta < 2\delta.$$

It remains to bound the difference of the integrals over  $\sim B_T$ . Since  $0 < \gamma(\sim B_T) < 2\delta$ ,  $\left| \int_{\sim B_T} g d\mu - \int_{\sim B_T} g d\gamma \right| < 2\delta + \delta$ . So  $\left| \int_{\Lambda} g d\mu - \int_{\Lambda} g d\gamma \right| < 5\delta = \varepsilon$ . ■

PROPOSITION A.6. Suppose  $\mu, \mu_i \in P_+(\Lambda)$  and  $\mu_i \Rightarrow \mu$ . Then  $\Gamma(\mu_i, z), \Gamma(\mu, z) \in P_+(\Lambda) \vee$  a.s., and  $\Gamma(\mu_i, \cdot) \rightarrow \Gamma(\mu, \cdot)$  in  $v$ -measure.

*Proof.* Verification that  $\Gamma(\mu_i, z), \Gamma(\mu, z) \in P_+(\Lambda)$  is routine and so omitted. Pick  $\varepsilon > 0$ , and as in Lemma A.5, let  $\mathcal{A} = \{A_1, A_2, \dots\}$  be a disjoint cover of  $\Lambda$  with diameter of  $A_t < \delta = \frac{\varepsilon}{5}$  and  $\mu(\partial A_t) = 0$ . (The existence of such a collection follows from Theorem 11.7.3. of Dudley (1989).) Define  $B_t$  as in Lemma A.5, and choose  $T$  s.t.  $\mu(B_T) > 1 - \delta$ . Observe that  $\vee$  a.s.,  $\Gamma(\mu_i, z)(A_t) = \frac{L_i(A_t, z)}{L_i(z)}$ , which is well-defined since  $\mu_i \in P_+(\Lambda)$  and so  $L_i(z) > 0$ .

Applying Lemmas A.3 and A.4, we have  $\frac{L_i(A_i, \cdot)}{L_i(\cdot)} \rightarrow \frac{L(A_i, \cdot)}{L(\cdot)}$  in  $\nu$ -measure. Since  $B_T$  is the union of a finite number of  $A_i$  sets, by Lemma A.5, we have  $\limsup_i \nu(\{z: \beta(\Gamma(\mu_i, z), \Gamma(\mu, z)) \geq \varepsilon\}) = 0$ . Since  $\varepsilon$  is arbitrary,  $\nu(Z, \beta(\Gamma(\mu_i, z), \Gamma(\mu, z))) \rightarrow 0$ . ■

**PROPOSITION A.7.** Suppose  $\mu, \mu_i \in P_+(\Lambda)$ ,  $\mu_i \Rightarrow \mu$ ,  $f: \Lambda \rightarrow \mathbf{R}$  is bounded and continuous, and  $\lambda_i \rightarrow \lambda \in (\Lambda, d_1)$ . Then 
$$\int_Z \int_{\Lambda} [f(\lambda) \Gamma(\mu_i, z)(d\lambda)] h(\lambda_i, z) \nu(dz) \rightarrow \int_Z \int_{\Lambda} [f(\lambda) \Gamma(\mu, z)(d\lambda)] h(\lambda, z) \nu(dz).$$

*Proof.* Define  $\varphi_i, \varphi: Z \rightarrow \mathbf{R}$  by  $\varphi_i(z) = \int_{\Lambda} f(\lambda) \Gamma(\mu_i, z)(d\lambda)$  and  $\varphi(z) = \int_{\Lambda} f(\lambda) \Gamma(\mu, z)(d\lambda)$ . By Proposition A.6,  $\varphi_i \rightarrow \varphi$  in  $\nu$ -measure. Since  $|\varphi_i|$  and  $|\varphi|$  are uniformly bounded by  $\|f\|$ , and  $h(\lambda_i, \cdot) \rightarrow h(\lambda, \cdot)$  in  $L^1(Z, B(Z), \nu)$ ,  $\varphi_i \cdot h(\lambda_i, \cdot) \rightarrow \varphi \cdot h(\lambda, \cdot)$  in  $L^1(Z, B(Z), \nu)$ . So 
$$\int_Z \varphi_i(z) \cdot h(\lambda_i, z) \nu(dz) \rightarrow \int_Z \varphi(z) \cdot h(\lambda, z) \nu(dz),$$
 or equivalently,

$$\int_Z \int_{\Lambda} [f(\lambda) \Gamma(\mu_i, z)(d\lambda)] h(\lambda_i, z) \nu(dz) \rightarrow \int_Z \int_{\Lambda} [f(\lambda) \Gamma(\mu, z)(d\lambda)] h(\lambda, z) \nu(dz). \quad \blacksquare$$

*Proof of PROPOSITION 3.5.* Fix  $k$  and  $m$ . For  $n = 1$ , setting  $f = g_{kmn}$ , the result follows directly from Proposition A.7. For  $n > 1$ , repeat the above arguments replacing  $z$  with  $\sigma$  and  $\nu$  with  $\nu^n$ . ■

## FOOTNOTES

<sup>1</sup>Some authors allow for heterogeneous beliefs across agents.

<sup>2</sup>The definition of an optimal plan when  $\alpha = 0$  will be provided in the next subsection.

<sup>3</sup>I am indebted to Neil Wallace for this observation.

<sup>4</sup>The above remarks are also applicable to nonparametric Bayesian econometrics. If the reader of a Bayesian statistical analysis has a prior not absolutely continuous with respect to the author's prior (and the prior is not consistent for all parameter values), even with large samples there may be no merging of posterior beliefs. For a more complete discussion of these issues is provided by Diaconis and Freedman (1986).

<sup>5</sup>Barron (1988) has recently proven results for stationary stochastic processes. Barron's techniques are potentially extendable to processes where endogenous learning generates the non-stationarity.

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